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Performance of Projection Methods for Low-Reynolds-Number Flows

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Resumo

The Navier-Stokes equations, modelling incompressible viscous Newtonian flows, can be written in non-dimensional form as

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \nabla\cdot(\mathbf{u}\mathbf{u}) - \frac{1}{Re}\nabla^2\mathbf{u} + \nabla p = \mathbf{f} \quad &\text{in} \quad [0,T]\times\Omega, \\ \nabla\cdot\mathbf{u} = 0 \quad &\text{in} \quad [0,T]\times\Omega, \end{aligned} \tag{1}$$

where t is time, $\mathbf{u}$ is the velocity vector field, p is the pressure scalar field. The non-dimensional quantity Re is known as Reynolds number, and represents the relation between kinematic and viscous forces, characterising the fluid flow. Additionally, $\mathbf{f}$ is a body-force that may be acting on the fluid. All these quantities are defined in a closed domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) for $t \in [0, T]$.

It is a well known fact that the system (1) has a strong coupling between velocity and pressure variables, and to avoid the solution of large coupled systems, a class of "projection" or "fractional step" methods were early designed [1, 2]. The basic idea behind these methods is to compute a tentative velocity field by using the momentum equation (1), which does not generally satisfy the continuity equation. Then, by using the Helmholtz-Hodge decomposition theorem, the intermediate velocity is projected to a divergence free subspace, to finally produce a solenoidal velocity field. In the last decades, since the introduction of the concept of projection method at the end of 1960s, many researchers have made a tremendous effort to extend, analyse and implement variations of the projection method, with special interest in obtaining second-order accuracy in time [3, 4, 5, 6]. In the classical approach, the splitting is first performed on the continuous differential equations, before any spatial

discretization, leading to a set of differential equations to be solved sequentially. In order to avoid artificial boundary condition issues with this methodology, algebraic splitting methods were designed [7, 8]. Different than the classical approach, the splitting is performed after spacial and temporal discretizations, decoupling the resulting algebraic system by using suitable matrix decompositions techniques [9].

With the growing interest in modelling flows in millimetric and micrometric scales, numerical methods have to be adapted to new posed challenges. Usually, this type of flows are characterised by low Reynolds numbers ($Re \ll 1$), which requires implicit time discretization schemes. In this work, we investigate the performance of projection methods (of the algebraic-splitting kind) for the computation of steady-state simple benchmark problems. The most popular approximate factorization methods are assessed, together with two so-called exact factorization methods. The results show that: (a) The error introduced by non-incremental schemes on the steady state solution is unacceptably large even in the simplest of flows. (b) Incremental schemes have an optimal time step δt^* so as to reach the steady state with minimum computational effort. Taking $\delta t = \delta t^*$ the code reaches the steady state in not less than a few hundred time steps. Such a cost is significantly higher than that of solving the velocity-pressure coupled system, which can compute the steady state in one shot. (c) If δt is chosen too large (in general δt^* is not known), then thousands or tens of thousands of time steps are required to reach the numerical steady state with incremental projection methods. The numerical solutions of these methods follow a time-step-dependent spurious transient which makes the computation of steady states prohibitively expensive.

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